AutLinOrd

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Chapter 1

Bumps, blocks, and bubbles

Bubbles

Our goal here is to define the bubble relation on a linear order. Let X be a linear order.

Definition 1. Let $f\colon X \to X$ an automorphism. For any $x \in X$, the **orbit** of x under f is the set $\{f^n(x) : n \in \mathbb{Z}\}$.

Definition 2. Let $f\colon X \to X$ an automorphism. For any $x \in X$, the **orbital** of x under f is the convex closure of the orbit of x under f.

Definition 3. A **bump** is an automorphism $f\colon X \to X$ with exactly one non-singleton orbital.

Definition 4. A **bounded bump** is a bump whose unique non-singleton orbital is either strictly bounded below or strictly bounded above.

Definition 5. We define a relation $\sim_b$ on X as follows: $x \sim_b y$ if and only if there exists a bounded bump such that x and y are in its orbital or $x = y$.

Theorem 6. *The relation $\sim_b$ is reflexive and symmetric.*

Proof. The definition is clearly reflexive and the definition is symmetric in x and y. □

Theorem 7. *The relation $\sim_b$ is convex.*

Proof. Let $x \in X$. We want to show that the $\sim_b$-equivalence class of x is an interval. Let y and z be elements of X such that $x < y < z$ and $x \sim_b z$. It cannot be that $x = z$ and so we must have that $x, z \in O(f)$ for some bounded bump $f\colon X \to X$. Since $O(f)$ is an interval and $x < y < z$, we have that $y \in O(f)$. Thus, $x \sim_b y$ and so we are done. □

Theorem 8. *The relation $\sim_b$ is transitive.*

Proof. Let x, y, and z be elements of X such that $x \sim_b y$ and $y \sim_b z$. If $x = y$ or $y = z$, then we are done. □