OrderedSemigroups

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Chapter 1

Groups

In this chapter, we prove Holder's theorem for groups. We follow the proof in "Groups, Orders, and Dynamics" by Deroin, Navas, and Rivas.

We choose an element of the ordered group f and map it to 1 in the real numbers. Then for any other element g of the ordered group, we use f to construct a sequence of rational approximations of g. We map g to the real number that is the limit of this sequence of approximations. We then prove that this map is injective and order preserving.

1.1 Definitions

We begin with basic definitions of ordered groups.

Definition 1. A **left ordered group** G is a group and a partial order such that for all $x, y, z \in G$, if $x \leq y$, then $z * x \leq z * y$.

Definition 2. A **left linear ordered group** G is a left ordered group that is also a linear order.

Definition 3. An **Archimedean group** is a left ordered group such that for any $g, h \in G$ where $g \neq 1$, there exists an integer z such that $h < g^z$.

1.2 Approximation

In this section we assume that G is a left linear ordered group that is Archimedean. Furthermore, we assume we have an element $f \in G$ such that $1 < f$.

Theorem 4. *For any $g \in G$ and $p \in \mathbb{N}$, there exists an integer $q \in \mathbb{Z}$ such that*

$$f^q \leq g^p < f^{q+1}$$

Proof. Since G is an Archimedean group, we can construct exponents l and u such that $f^l < g^p < f^u$. Therefore, there must exist some integer q which satisfies what we want. □

Definition 5. We define a function $q\colon G \to \mathbb{N} \to \mathbb{R}$ using Theorem 4 such that for any $g \in G$ and $n \in \mathbb{N}$,

$$f^{q_g(n)} \leq g^n < f^{q_g(n)+1}$$

Theorem 6. *For any sequence a_n of real numbers, if there exists $C \in \mathbb{R}$ such that for all $m, n \in \mathbb{N}$ we have that*

$$|a_{m+n} - a_m - a_n| \leq C$$

then sequence $\frac{a_n}{n}$ converges.

Proof. Not included here as the ideas are separate from this project. $\square$

Theorem 7. *For any $g \in G$ and $a, b \in \mathbb{N}$, we have that*

$$f^{q_g(a)+q_g(b)} \leq g^{a+b} < f^{q_g(a)+q_g(b)+2}$$

Proof. We know the following two things by the definition of q

$$\begin{aligned} f^{q_g(a)} &\leq g^a < f^{q_g(a)+1} \\ f^{q_g(b)} &\leq g^b < f^{q_g(b)+1} \end{aligned}$$

And so it follows that

$$f^{q_g(a)+q_g(b)} \leq g^{a+b} < f^{q_g(a)+q_g(b)+2}$$

$\square$

Theorem 8. *For any $g \in G$, the sequence $\frac{q_g(n)}{n}$ converges.*

Proof. From Theorem 7, we have that

$$q_g(a) + q_g(b) \leq q_g(a+b) \leq q_g(a) + q_g(b) + 1$$

and so

$$|q_g(a+b) - q_g(a) - q_g(b)| \leq 1$$

Therefore, by Theorem 6 with $C = 1$, we have that the sequence $\frac{q_g(n)}{n}$ converges. $\square$

1.3 Map

We make the same assumptions as in the previous section. So we assume that G is a left linear ordered group that is Archimedean, $f \in G$, and $1 < f$.

We now define the map from the G to $\mathbb{R}$ and prove its properties.

Definition 9. We define a map $\phi \colon G \to \mathbb{R}$ by mapping g to the real number that $\frac{q_g(n)}{n}$ converges to as we know from Theorem 8.

Theorem 10. *For all $g_1, g_2 \in G$ and $p \in \mathbb{N}$,*

$$q_{g_1}(p) + q_{g_2}(p) \leq q_{g_1 g_2}(p) \leq q_{g_1}(p) + q_{g_2}(p) + 1$$

Proof. Let $q_1 = q_{g_1}(p)$ and $q_2 = q_{g_2}(p)$. Then we know that

$$\begin{aligned} f^{q_1} &\leq g_1^p < f^{q_1+1} \\ f^{q_2} &\leq g_2^p < f^{q_2+1} \end{aligned}$$

And so we also have the following two facts

$$f^{q_1+q_2} \leq g_1^p g_2^p$$
$$g_2^p g_1^p < f^{q_1+q_2+2}$$

We look at the case where $g_1 g_2 \leq g_2 g_1$. Then $g_1^p g_2^p \leq (g_1 g_2)^p \leq g_2^p g_1^p$. And so combined with the previous facts, we have that

$$f^{q_1+q_2} \leq g_1^p g_2^p \leq (g_1 g_2)^p \leq g_2^p g_1^p < f^{q_1+q_2+2}$$

Therefore,

$$q_1 + q_2 \leq q_{g_1 g_2}(p) \leq q_1 + q_2 + 1$$

And so we are done. The case where $g_2 g_1 \leq g_1 g_2$ follows similarly. □

Theorem 11. *The map ϕ is a homomorphism.*

Proof. Let $g_1, g_2 \in G$. Then from Theorem 10 we have that

$$q_{g_1}(p) + q_{g_2}(p) \leq q_{g_1 g_2}(p) \leq q_{g_1}(p) + q_{g_2}(p) + 1$$

And so since $\lim_{p\to\infty} \frac{q_{g_1}(p)+q_{g_2}(p)}{p} = \lim_{p\to\infty} \frac{q_{g_1}(p)+q_{g_2}(p)+1}{p}$, we have that

$$\lim_{p\to\infty} \frac{q_{g_1}(p) + q_{g_2}(p)}{p} = \lim_{p\to\infty} \frac{q_{g_1 g_2(p)}}{p}$$

Therefore, by the definition of ϕ, we have shown that $\phi(g_1) + \phi(g_2) = \phi(g_1 g_2)$. □

Theorem 12. *For all $g, h \in G$, if $g \leq h$ then $\phi(g) \leq \phi(h)$.*

Proof. First, notice that since $g \leq h$, then for all $p \in \mathbb{N}$, $q_g(p) \leq q_h(p)$. Then from the definition of ϕ, it follows that $\phi(g) \leq \phi(h)$. □

Theorem 13. *We have that $\phi(f) = 1$ where f is our fixed positive element.*

Proof. We have that for all $n \in \mathbb{n}$ that $f^n \leq f^n < f^{n+1}$ and so $q_f(n) = n$. Therefore, $\phi(f) = 1$. □

Theorem 14. *The map ϕ is injective.*

Proof. Since from Theorem 11 we have that ϕ is a homomorphism, it suffices to show that for any $g \in G$, if $\phi(g) = 0$, then $g = 1$.

Assume for the sake of contradiction that there exists $g \in G$ such that $\phi(g) = 0$ but g is not equal to 1. Then since G is Archimedean, there exists an integer z such that $g^z > f$. Therefore, since by Theorem 13 we have that $\phi(f) = 1$,

$$0 = \phi(g) = \phi(g)^z = \phi(g^z)$$
$$> \phi(f) = 1$$

Contradiction. □

Theorem 15. *For all $g, h \in G$, we have that $g \leq h$ if and only if $\phi(g) \leq \phi(h)$.*

Proof. ($\Rightarrow$) This is Theorem 12.

($\Leftarrow$) We have that $\phi(g) \leq \phi(h)$. Assume for the sake of contradiction that $h < g$. Then by Theorem 12, we know that $\phi(h) \leq \phi(g)$. Therefore, $\phi(g) = \phi(h)$. And so by Theorem 14, we know that ϕ is injective and so $g = h$. Contradiction. □

1.4 Holder's Theorem

Theorem 16. *If G is a left linear ordered group that is Archimedean, then G is isomorphic to a subgroup of $\mathbb{R}$.*

Proof. First, we look at the case where there exists a positive element f in G. Given such an element, we have an order preserving, injective homomorphism ϕ. And so G is isomorphic to the image of ϕ which is a subgroup of $\mathbb{R}$.

If there does not exist a positive element in G, then G is trivial and is isomorphic to the trivial subgroup of $\mathbb{R}$. $\square$

Chapter 2

Semigroups

We follow the paper "On ordered semigroups" by N. G. Alimov.

We show that if a a linear ordered cancel semigroup does not have anomalous pairs, then there exists a larger Archimedean group containing it. Then from Holder's theorem for groups, that larger group is isomorphic to a subgroup of $\mathbb{R}$ and so the smaller semigroup is isomorphic to a subsemigroup of $\mathbb{R}$.

2.1 Definitions

Definition 17. A **left ordered semigroup** S is a semigroup and a partial order such that for all $x, y, z \in S$, if $x \leq y$, then $z * x \leq z * y$.

Definition 18. A **right ordered semigroup** S is a semigroup and a partial order such that for all $x, y, z \in S$, if $x \leq y$, then $x * z \leq y * z$.

Definition 19. An **ordered semigroup** S is a left and right ordered semigroup.

Definition 20. An **ordered cancel semigroup** S is an ordered semigroup such that for all $a, b, c \in S$, if $a * b \leq a * c$ then $b \leq c$ and if $b * a \leq c * a$ then $b \leq c$.

Definition 21. A **linear ordered semigroup** is an ordered semigroup where its partial order is a linear order.

Definition 22. A **linear ordered cancel semigroup** is an ordered cancel semigroup where its partial order is a linear order.

Definition 23. An **anomalous pair** in a left ordered semigroup S is a pair of elements $a, b \in S$ such that for all positive $n \in \mathbb{N}$, either

$$a^n < b^n < a^{n+1}$$

or

$$a^n > b^n > a^{n+1}$$

.

Intuitively, an anomalous pair represents a pair of elements that are infinitely close. If you have two real numbers s and r such that $s < r$, then as $n \in \mathbb{N}$ gets larger, $n \cdot s$ and $n \cdot r$ get farther apart. However, for an anomalous pair, the elements always remain close to each other.

Definition 24. An element a of a left ordered semigroup S is **positive** if for all $x \in S$, $a*x > x$.

Definition 25. An element a of a left ordered semigroup S is **negative** if for all $x \in S$, $a*x < x$.

Definition 26. An element a of a left ordered semigroup S is **one** if for all $x \in S$, $a * x = x$.

Definition 27. Let a and b be elements of a left ordered semigroup S that are not one.

Then a is said to be **Archimedean with respect to** b if there exists an $N \in \mathbb{N}^+$ such that for $n > N$, the inequality $b < a^n$ holds if b is positive, and the inequality $b > a^n$ holds if b is negative.

Definition 28. A left ordered semigroup is **Archimedean** if any two of its elements of the same sign are mutually Archimedean.

Definition 29. A left ordered semigroup S has **large differences** if for all $a, b \in S$, the two following implications hold

- if a is positive and $a < b$, then there exists $c \in S$ and $n \in \mathbb{N}^+$ such that c is Archimedean with respect to a and $a^n * c \leq b^n$
- if a is negative and $b < a$, then there exists $c \in S$ and $n \in \mathbb{N}^+$ such that c is Archimedean with respect to a and $a^n * c \geq b^n$

Intuitively, this is saying if $a < b$ then eventually a^n is significantly smaller than b^n. Here "significantly smaller" means that there is an element that is not infinitely small with respect to a that separates a^n and b.

2.2 Anomalous Pairs

Theorem 30. *Each element a of a linear ordered cancel semigroup S is either positive, negative, or one.*

Proof. Let $a \in S$ and $b \in S$. Since the S is a linear order we have one of the following cases.

The first case is that $b * a > b$. Then for all $x \in S$ we have that $b * a * x > b * x$ and so $a * x > x$. Therefore, a is positive.

The second case is that $b * a < b$. Then for all $x \in S$ we have that $b * a * x < b * x$ and so $a * x < x$. Therefore, a is negative.

The last case is that $b * a = b$. Then for all $x \in S$ we have that $b * a * x = b * x$ and so $a * x = x$. Therefore, a is zero. ☐

Theorem 31. *If S is a non-Archimedean linear ordered cancel semigroup, then there exists an anomalous pair.*

Proof. Since S is non-Archimedean, there exists $a, b \in S$ such that a and b have the sign and are not mutually Archimedean.

First, we look at the case where a and b are positive. Then since they are not mutually Archimedean, without loss of generality, for all $n \in \mathbb{N}^+$, $a^n < b$.

Then we either have that $a * b \leq b * a$ or that $b * a \leq a * b$. In the first case, we have that

$$a^n + b^n \leq (a * b)^n \leq b^n + a^n$$

which means that, since $a^n < b$,

$$b^n < (a * b)^n < b^{n+1}$$

And so b and $a * b$ form an anomalous pair.

The other cases follow similarly. ☐

Theorem 32. *A linear ordered cancel semigroup without anomalous pairs is an Archimedean commutative semigroup.*

Proof. Let a and b be elements of an ordered semigroup S. If either a or b is one, then they commute.

We begin with the case that a and b are positive. If $a * b < b * a$, then for all $n \in \mathbb{N}^+$, we have that

$$\begin{aligned}(a * b)^{n+1} &= a * (b * a)^n * b \\ &> (b * a)^n * b \\ &> (b * a)^n \\ &> (a * b)^n\end{aligned}$$

And so $a * b$ and $b * a$ form an anomalous pair.

The same for the case that $b * a < a * b$. Therefore, we must have that $a * b = b * a$.

The case where a and b are negative follows similarly.

We now look at the case where a is negative and b is positive. If the element 1 exists and $a * b = 1$, then $a * b * a = a$ and so $b * a = 1$. Therefore, a and b commute.

If $a * b$ is positive, then

$$\begin{aligned}(a * b) * (a * b) &> a * b \\ (b * a) * b &> b \\ b * a &> 0\end{aligned}$$

We already showed that positive elements commute and so

$$(b * a) * b = b * (b * a)$$

Now we look at the case where $a * b < b * a$. Then we have that

$$\begin{aligned}(a * b)^2 &= a * ((b * a) * b) \\ &= a * (b * (b * a)) \\ &= (a * b) * (b * a) \\ &> (a * b) * (a * b) \\ &= (a * b)^2\end{aligned}$$

Which is a contradiction. We get a similar contradiction in the case that $b * a < a * b$. Therefore, $a * b = b * a$.

The case where $a * b$ is negative follows similarly. The case where b is negative and a is positive is symmetric. □

Theorem 33. *In a linear ordered cancel semigroup S, there are no anomalous pairs if and only if there are large differences.*

Proof. (⇒) If a and b are positive and $a < b$, then we have that $a^n < b^n$ for all n. Therefore, there must exist an n such that $a^{n+1} \leq b^n$ as otherwise a and b would form an anomalous pair. Thus, the theorem is satisfied with $c = a$.

Similarly if a and b are negatiave.

(⇐) We look first at the case where a and b are positive and $a < b$. Then we have $c \in S$ Archimedean with respect to a and $m \in \mathbb{N}^+$ such that $a^m * c \leq b^m$.

Therefore, for any $n \in \mathbb{N}^+$, either

$$(a^m)^n * c^n \leq (a^m * c)^n \leq (b^m)^n$$

or

$$c^n * (a^m)^n \leq (a^m * c)^n \leq (b^m)^n$$

holds.

Since c is Archimedean with respect to a, there exists an N such that for all $n \geq N$, $a < c^n$. Thus, for any $n \geq N$, we get from the previous relations that

$$a^{mn+1} \leq b^{mn}$$

and so a and b do not form an anomalous pair.

Similalry if a and b are negative. □

2.3 Semigroup to Group

Theorem 34. *If S is a linear ordered cancel semigroup without anomalous pairs, then there exists a linear ordered cancel commutative monoid M without anomalous pairs such that S is isomorphic to a subsemigroup of M.*

Proof. We do casework on whether or not S has an element that is one.

If S has an element that is one then it is already a monoid. Furthermore, since it has no anomalous pairs, by Theorem 32, it is commutative. And so we are done.

If S does not have an element that is one, then we let M be S with one added. We define one to be less than every positive element and greater than every negative element. By Theorem 32, we know that S is commutative and so M is too. Furthermore, it is clear that M has no anomalous pairs still. Then S is isomorphic to the subsemigroup of M that is all its elements except for the added one. □

Theorem 35. *Let S be a linear ordered cancel semigroup without anomalous pairs such that there exists a positive element of S. Then for all $x \in S$, there exists a $y \in S$ such that y is positive and $x * y$ is positive.*

Similarly, if there exists a negative element of S then for all $x \in S$ there exists a $y \in S$ such that y is negative and $x * y$ is negative.

Proof. We look at the positive case. If x is positive then we are done as $x * x$ is positive.

Next we look at the case where x is negative. Assume for the sake of contradiction that for all $y \in S$, if y is positive then $x * y$ is negative. Let z be a positive element we assumed existed in S. Then for all positive $n \in \mathbb{N}$, we have that $x * z^{n+2}$ is negative. Recall that from Theorem 32 we have commutativity. From there we have an anomalous pair:

$$\begin{aligned}(x * z)^n &= x^n * z^n \geq x^n \\ &\geq x^n * (x * z^{n+2}) \\ &= (x * z)^{n+1} * z > (x * z)^{n+1}\end{aligned}$$

Contradiction. And similarly for the negative case. □

Theorem 36. *Let M be a linear ordered cancel commutative monoid without anomalous pairs and let G be its Grothendieck group. If M has a positive (negative) element, then for any element $\frac{a}{b} \in G$ where $a, b \in M$, there exist $a', b' \in M$ such that a' and b' are positive (negative) and $\frac{a}{b} = \frac{a'}{b'}$.*

Proof. Let $\frac{a}{b} \in G$ where $a, b \in M$. Since M has a positive element, by Theorem 35, we have $a_2, b_2 \in M$ such that $a * a_2$ and $b * b_2$ are positive. Let $a' = a * a_2$ and $b' = b * b_2$. Then $\frac{a}{b} = \frac{a'}{b'}$ and a' and b' are positive.

And similarly for the negative case. $\square$

Theorem 37. *Let M be a linear ordered cancel commutative monoid. If M does not have anomalous pairs, then its Grothendieck group is Archimedean.*

Proof. If M is trivial then we are done. Otherwise, it has a positive element of a negative element.

We look at the case where M has a positive element t. We now want to show that the Grothendieck group G of M is Archimedean. It suffices to show that for any positive $g, h \in G$, there exists an integer n such that $g^n > h$. Since $g, h \in G$, there exist $g_1, g_2, h_1, h_2 \in M$ such that $g = \frac{g_1}{g_2}$ and $h = \frac{h_1}{h_2}$, and by Theorem 36, we can assume that g_1, g_2, h_1, and h_2 are positive.

Then since M does not have anomalous pairs, we have by Theorem 32 that M is Archimedean. Since g_1 and h_1 are positive and M is Archimedean, there exists a positive $N \in \mathbb{N}$ such that for all $n \geq N$, $g_2^n > h_1$. So in particular, we have that $g_2^N > h_1$.

Notice that since $\frac{g_1}{g_2}$ positive, we have that $g_2 < g_1$. And therefore, since M does not have anomalous pairs, there exists a positive natural number N_1 such that $g_2^{N_1+N} < g_1^{N_1}$.

We now claim that $g^{N_1} > h$. This is equivalent to showing that $g_2^{N_1} * h_1 < h_2 * g_1^{N_1}$. And we have that

$$\begin{aligned} g_2^{N_1} * h_1 &= h_1 * g_2^{N_1} \\ &< g_2^N * g_2^{N_1} = g_2^{N_1+N} \\ &< g_1^{N_1} \\ &< h_1 * g_1^{N_1} \end{aligned}$$

And so we are done.

The final case where M has a negative element follows similarly. $\square$

Theorem 38. *If S is a linear ordered cancel semigroup that does not have anomalous pairs, then there exists a linear ordered commutative group G that is Archimedean such that S is isomorphic to a subsemigroup of G.*

Proof. By Theorem 34, we have that S is isomorphic to a subsemigroup of a linear ordered cancel commutative monoid M that does not have anomalous pairs.

We let G be the Grothendieck group of M. Then by Theorem 37 we know that G is Archimedean. Since G is the Grothendieck group of M, M is isomorphic to a submonoid of G. Thus, we have that S is isomorphic to a subsemigroup of G. $\square$

2.4 Holder's Theorem

Theorem 39. *Let S be a linear ordered cancel semigroup without anomalous pairs. Then S is isomorphic to a subsemigroup of the real numbers.*

Proof. By Theorem 38, there exists a linear ordered commutative Archimedean group G such that S is isomorphic to a subsemigroup of G. By Theorem 16, G is isomorphic to a subgroup of $\mathbb{R}$. Therefore, S is isomorphic to a subsemigroup of $\mathbb{R}$. $\square$